

# *Algebraic Structure of a Master Equation with Generalized Lindblad Form*

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## **Abstract**

The quantum damped harmonic oscillator is described by the master equation with usual Lindblad form. The equation has been solved completely by us in arXiv : 0710.2724 [quant-ph]. To construct the general solution a few facts of representation theory based on the Lie algebra  $su(1, 1)$  were used.

In this paper we treat a general model described by a master equation with generalized Lindblad form. Then we examine the algebraic structure related to some Lie algebras and construct the interesting approximate solution.

Quantum Computation (Computer) is one of main subjects in Quantum Physics. To realize it we must overcome severe problems arising from Decoherence, so we need to study Quantum Open System to control decoherence (if possible).

This paper is a series of [1] and [2], and we study dynamics of a quantum open system. First we explain our purpose in a short manner. See [3] as a general introduction to this subject.

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We consider a quantum open system  $S$  coupled to the environment  $E$ . Then the total system  $S + E$  is described by the Hamiltonian

$$H_{S+E} = H_S \otimes \mathbf{1}_E + \mathbf{1}_S \otimes H_E + H_I$$

where  $H_S$ ,  $H_E$  are respectively the Hamiltonians of the system and environment, and  $H_I$  is the Hamiltonian of the interaction.

Then under several assumptions (see [3]) the reduced dynamics of the system (which is not unitary !) is given by the Master Equation

$$\frac{\partial}{\partial t} \rho = -i[H_S, \rho] - \mathcal{D}(\rho) \quad (1)$$

with the dissipator being the usual Lindblad form

$$\mathcal{D}(\rho) = \frac{1}{2} \sum_{\{j\}} \left( A_j^\dagger A_j \rho + \rho A_j^\dagger A_j - 2A_j \rho A_j^\dagger \right). \quad (2)$$

Here  $\rho \equiv \rho(t)$  is the density operator (or matrix) of the system.

Similarly, the equation of quantum damped harmonic oscillator (see [3], Section 3.4.6) is given by

$$\frac{\partial}{\partial t} \rho = -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a), \quad (3)$$

where  $a$  and  $a^\dagger$  are the annihilation and creation operators of the system (for example, an electro-magnetic field mode in a cavity), and  $\mu$ ,  $\nu$  are some real constants depending on the system (for example, a damping rate of the cavity mode).

Since this is one of fundamental equations in quantum theory it is very important to construct the general solution. In [3] or [4] some methods to construct a solution are presented. However, in [2] we gave the general solution in **the operator algebra level**. This is a very important step.

Our method is as follows : we clarified a certain algebraic structure arising from the Lie algebra  $su(1,1)$  and its representation in the equation (3) and constructed the general solution by use of the (well-known) disentangling formula. See for example [5] and [6]. The method is popular in Quantum Optics, while it may be not in the field of Quantum Open System.

In this paper we want to generalize the model in order to examine deeper algebraic structures. Namely, we consider a master equation with generalized Lindblad form defined by

$$\begin{aligned} \frac{\partial}{\partial t}\rho = & -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a) \\ & - \frac{\kappa}{2} (a^2 \rho + \rho a^2 - 2a \rho a) - \frac{\bar{\kappa}}{2} ((a^\dagger)^2 \rho + \rho (a^\dagger)^2 - 2a^\dagger \rho a^\dagger) \end{aligned} \quad (4)$$

where  $\kappa$  is a complex constant satisfying the condition  $\mu\nu \geq |\kappa|^2$  which ensures the positivity. See for example [7]<sup>1</sup>.

Then we examine an algebraic structure related to the Lie algebras  $su(1, 1)$  and  $su(2)$ , and construct interesting approximate solutions by use of it.

In order to solve the equation we use the method in [1] once more. For that we review a matrix representation of  $a$  and  $a^\dagger$  on the usual Fock space

$$\mathcal{F} = \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots\}; \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

like

$$a = e^{i\theta} \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}, \quad a^\dagger = e^{-i\theta} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{pmatrix} \quad (5)$$

$$N = a^\dagger a = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} \quad (6)$$

where  $e^{i\theta}$  is some phase. Note that  $aa^\dagger = a^\dagger a + 1 = N + 1$ .

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<sup>1</sup>In the paper it is called the Kossakowski–Lindblad form not the generalized Lindblad one. It may be suitable.

For a matrix  $X = (x_{ij}) \in M(\mathcal{F})$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we correspond to the vector  $\widehat{X} \in \mathcal{F}^{\dim_{\mathbf{C}} \mathcal{F}}$  as

$$X = (x_{ij}) \longrightarrow \widehat{X} = (x_{11}, x_{12}, x_{13}, \cdots; x_{21}, x_{22}, x_{23}, \cdots; x_{31}, x_{32}, x_{33}, \cdots; \cdots)^T \quad (7)$$

where  $T$  means the transpose. The following formula

$$\widehat{AXB} = (A \otimes B^T) \widehat{X} \quad (8)$$

holds for  $A, B, X \in M(\mathcal{F})$ .

Then (4) is transformed into

$$\frac{\partial}{\partial t} \widehat{\rho}(t) = \widehat{H} \widehat{\rho}(t) \implies \widehat{\rho}(t) = e^{t\widehat{H}} \widehat{\rho}(0) \quad (9)$$

where

$$\begin{aligned} \widehat{H} = & -i\omega(N \otimes \mathbf{1} - \mathbf{1} \otimes N) \\ & -\frac{\mu}{2}\{N \otimes \mathbf{1} + \mathbf{1} \otimes N - 2a \otimes (a^\dagger)^T\} - \frac{\nu}{2}\{(N+1) \otimes \mathbf{1} + \mathbf{1} \otimes (N+1) - 2a^\dagger \otimes a^T\} \\ & -\frac{\kappa}{2}\{a^2 \otimes \mathbf{1} + \mathbf{1} \otimes (a^2)^T - 2a \otimes a^T\} - \frac{\bar{\kappa}}{2}\{(a^\dagger)^2 \otimes \mathbf{1} + \mathbf{1} \otimes ((a^\dagger)^2)^T - 2a^\dagger \otimes (a^\dagger)^T\}. \end{aligned} \quad (10)$$

Moreover it is rewritten as

$$\begin{aligned} \widehat{H} = & \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1} - (\mu + \nu) \frac{N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}}{2} + \nu a^\dagger \otimes a^T + \mu a \otimes (a^\dagger)^T \\ & - 2i\omega \frac{N \otimes \mathbf{1} - \mathbf{1} \otimes N}{2} + \bar{\kappa} \left\{ a^\dagger \otimes (a^\dagger)^T - \frac{(a^\dagger)^2 \otimes \mathbf{1} + \mathbf{1} \otimes ((a^\dagger)^2)^T}{2} \right\} \\ & + \kappa \left\{ a \otimes a^T - \frac{a^2 \otimes \mathbf{1} + \mathbf{1} \otimes (a^2)^T}{2} \right\}. \end{aligned} \quad (11)$$

Now let us examine the algebraic structure of  $\widehat{H}$ .

By setting

$$\tilde{K}_3 = \frac{1}{2}(N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}), \quad \tilde{K}_+ = a^\dagger \otimes a^T, \quad \tilde{K}_- = a \otimes (a^\dagger)^T \quad (12)$$

where  $N^T = N$ , then we have

$$[\tilde{K}_3, \tilde{K}_+] = \tilde{K}_+, \quad [\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_-, \quad [\tilde{K}_+, \tilde{K}_-] = -2\tilde{K}_3. \quad (13)$$

Namely,  $\{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\}$  is a set of generators of  $su(1, 1)$  algebra, [2].

By setting

$$J_3 = \frac{1}{2}(N \otimes \mathbf{1} - \mathbf{1} \otimes N), \quad J_+ = a^\dagger \otimes (a^\dagger)^T, \quad J_- = a \otimes a^T \quad (14)$$

, then we have

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \quad (15)$$

Namely,  $\{J_3, J_+, J_-\}$  is a set of generators of  $su(2)$  algebra.

By setting

$$K_3 = \frac{1}{2}(N \otimes \mathbf{1} - \mathbf{1} \otimes N), \quad K_+ = \frac{1}{2}\{(a^\dagger)^2 \otimes \mathbf{1} + \mathbf{1} \otimes ((a^\dagger)^2)^T\}, \quad K_- = \frac{1}{2}\{a^2 \otimes \mathbf{1} + \mathbf{1} \otimes (a^2)^T\} \quad (16)$$

, then we have

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (17)$$

Namely,  $\{K_3, K_+, K_-\}$  is a set of generators of  $su(1, 1)$  algebra.

By setting

$$L_3 = \frac{1}{2}(N \otimes \mathbf{1} - \mathbf{1} \otimes N), \quad L_+ = J_+ - K_+, \quad L_- = J_- - K_- \quad (18)$$

, then we have

$$[L_3, L_+] = L_+, \quad [L_3, L_-] = -L_-, \quad [L_+, L_-] = 0. \quad (19)$$

We also note that

$$[J_+, K_+] = [J_-, K_-] = 0. \quad (20)$$

However,  $\{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\}$  and  $\{L_3, L_+, L_-\}$  don't commute except for

$$[L_3, \tilde{K}_3] = [L_3, \tilde{K}_+] = [L_3, \tilde{K}_-] = 0, \quad (21)$$

see [2].

A comment is in order. For later convenience let us write down the remaining commutators.

$$\begin{aligned} [\tilde{K}_3, L_+] &= -\frac{1}{2}\{(a^\dagger)^2 \otimes \mathbf{1} - \mathbf{1} \otimes ((a^\dagger)^2)^T\}, \quad [\tilde{K}_3, L_-] = \frac{1}{2}\{(a^2 \otimes \mathbf{1} - \mathbf{1} \otimes (a^2)^T\}, \\ [\tilde{K}_+, L_+] &= a^\dagger \otimes (a^\dagger)^T - (a^\dagger)^2 \otimes \mathbf{1}, \quad [\tilde{K}_+, L_-] = a \otimes a^T - \mathbf{1} \otimes (a^2)^T, \\ [\tilde{K}_-, L_+] &= -a^\dagger \otimes (a^\dagger)^T + \mathbf{1} \otimes ((a^\dagger)^2)^T, \quad [\tilde{K}_-, L_-] = -a \otimes a^T + a^2 \otimes \mathbf{1}. \end{aligned}$$

Then (11) is rewritten like

$$\hat{H} = \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1} - (\mu + \nu) \tilde{K}_3 + \nu \tilde{K}_+ + \mu \tilde{K}_- - 2i\omega L_3 + \bar{\kappa} L_+ + \kappa L_-. \quad (22)$$

What we want to do is to calculate the evolution operator  $e^{t\hat{H}}$ , which is in general not easy. Since  $\{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\}$  and  $\{L_3, L_+, L_-\}$  don't commute it is reasonable to assume

$$e^{t\hat{H}} \approx e^{\frac{\mu - \nu}{2}t} e^{t\{-(\mu + \nu)\tilde{K}_3 + \nu\tilde{K}_+ + \mu\tilde{K}_-\}} e^{t\{-2i\omega L_3 + \bar{\kappa} L_+ + \kappa L_-\}} \quad (23)$$

as **the first approximation**.

A comment is in order. In place of (23) it may be also reasonable to take

$$e^{t\hat{H}} \approx e^{\frac{\mu - \nu}{2}t} e^{t\bar{\kappa} L_+} e^{t\{-2i\omega L_3 - (\mu + \nu)\tilde{K}_3 + \nu\tilde{K}_+ + \mu\tilde{K}_-\}} e^{t\kappa L_-}.$$

However, we don't consider this approximation in the paper.

By the way, we have calculated the term  $e^{t\{-(\mu + \nu)\tilde{K}_3 + \nu\tilde{K}_+ + \mu\tilde{K}_-\}}$  in [2]. The result is

$$e^{t\{-(\mu + \nu)\tilde{K}_3 + \nu\tilde{K}_+ + \mu\tilde{K}_-\}} = e^{G(t)K_+} e^{-2\log(F(t))K_3} e^{E(t)K_-}, \quad (24)$$

or more explicitly

$$\begin{aligned} \text{RHS} &= \frac{1}{F(t)} \exp(G(t)a^\dagger \otimes a^T) \left( \exp(-\log(F(t))N) \otimes \exp(-\log(F(t))N)^T \right) \times \\ &\quad \exp(E(t)a \otimes (a^\dagger)^T) \end{aligned} \quad (25)$$

with

$$\begin{aligned} E(t) &= \frac{\frac{2\mu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right)}{\cosh\left(\frac{\mu - \nu}{2}t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right)}, \quad G(t) = \frac{\frac{2\nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right)}{\cosh\left(\frac{\mu - \nu}{2}t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right)} \\ F(t) &= \cosh\left(\frac{\mu - \nu}{2}t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right). \end{aligned} \quad (26)$$

On the other hand, we can calculate the term  $e^{t\{-2i\omega L_3 + \bar{\kappa}L_+ + \kappa L_-\}}$  easily because of the relation  $[L_+, L_-] = 0$  in (19). The result is

$$e^{t\{-2i\omega L_3 + \bar{\kappa}L_+ + \kappa L_-\}} = e^{f(t)L_+} e^{g(t)L_3} e^{l(t)L_-} = e^{f(t)(J_+ - K_+)} e^{g(t)L_3} e^{l(t)(J_- - K_-)} \quad (27)$$

with

$$f(t) = \frac{e^{-2i\omega t} - 1}{-2i\omega} \bar{\kappa}, \quad g(t) = -2i\omega t, \quad l(t) = \frac{e^{-2i\omega t} - 1}{-2i\omega} \kappa. \quad (28)$$

More explicitly

$$\begin{aligned} \text{RHS} = & \exp(f(t)a^\dagger \otimes (a^\dagger)^T) \left( \exp\left(-\frac{f(t)}{2}(a^\dagger)^2\right) \otimes \exp\left(-\frac{f(t)}{2}((a^\dagger)^2)^T\right) \right) \times \\ & \exp\left(\frac{g(t)}{2}N\right) \otimes \exp\left(-\frac{g(t)}{2}N^T\right) \times \\ & \exp(l(t)a \otimes a^T) \left( \exp\left(-\frac{l(t)}{2}a^2\right) \otimes \exp\left(-\frac{l(t)}{2}(a^2)^T\right) \right) \end{aligned} \quad (29)$$

because of (20).

Therefore our approximate solution is

$$\begin{aligned} \hat{\rho}(t) \approx & e^{\frac{\mu-\nu}{2}t} e^{t\{-(\mu+\nu)\tilde{K}_3 + \nu\tilde{K}_+ + \mu\tilde{K}_-\}} e^{t\{-2i\omega L_3 + \bar{\kappa}L_+ + \kappa L_-\}} \hat{\rho}(0) \\ = & \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \exp(G(t)a^\dagger \otimes a^T) \times \\ & \left( \exp(-\log(F(t))N) \otimes \exp(-\log(F(t))N^T) \right) \times \\ & \exp(E(t)a \otimes (a^\dagger)^T) \times \\ & \exp(f(t)a^\dagger \otimes (a^\dagger)^T) \left( \exp\left(-\frac{f(t)}{2}(a^\dagger)^2\right) \otimes \exp\left(-\frac{f(t)}{2}((a^\dagger)^2)^T\right) \right) \times \\ & \exp\left(\frac{g(t)}{2}N\right) \otimes \exp\left(-\frac{g(t)}{2}N^T\right) \times \\ & \exp(l(t)a \otimes a^T) \left( \exp\left(-\frac{l(t)}{2}a^2\right) \otimes \exp\left(-\frac{l(t)}{2}(a^2)^T\right) \right) \hat{\rho}(0) \end{aligned}$$

and we restore this form to the usual one by use of (8). The result is

$$\begin{aligned} \rho(t) = & \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-\log(F(t))\}) N \} \times \\ & \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \phi(t) (a^\dagger)^m \right\} \exp(\{-\log(F(t))\}) N \} a^n \end{aligned} \quad (30)$$

and

$$\begin{aligned} \phi(t) = & \sum_{k=0}^{\infty} \frac{f(t)^k}{k!} (a^\dagger)^k \exp\left(-\frac{f(t)}{2}(a^\dagger)^2\right) \left\{ \exp\left(\frac{g(t)}{2}N\right) \times \right. \\ & \left. \left\{ \sum_{j=0}^{\infty} \frac{l(t)^j}{j!} a^j \exp\left(-\frac{l(t)}{2}a^2\right) \rho(0) \exp\left(-\frac{l(t)}{2}a^2\right) a^j \right\} \exp\left(-\frac{g(t)}{2}N\right) \right\} \\ & \exp\left(-\frac{f(t)}{2}(a^\dagger)^2\right) (a^\dagger)^k. \end{aligned} \quad (31)$$

This is indeed complicated.

To construct the general solution to the equation (4) is very important in not only Physics but also Mathematics. However, it is not easy at the moment, so we only constructed some approximate solution. In the very near future we would like to do it.

In this paper we revisited the quantum damped harmonic oscillator with generalized Lindblad form and constructed some approximate solution in the operator algebra level.

The model is very important to understand several phenomena related to quantum open systems, so the general solution is required.

On the other hand we are studying some related topics from a different point of view, see [8] and [9].

Lastly, we conclude the paper by stating our motivation. We are studying a model of quantum computation (computer) based on Cavity QED (see [10] and [11]), so in order to construct a more realistic model of (robust) quantum computer we have to study severe problems coming from decoherence.

For example, we have to study the quantum damped Jaynes–Cummings model (in our terminology) whose phenomenological master equation for the density operator is given by

$$\frac{\partial}{\partial t} \rho = -i[H_{JC}, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a), \quad (32)$$



where  $H_{JC}$  is the well-known Jaynes-Cummings Hamiltonian given by

$$\begin{aligned} H_{JC} &= \frac{\omega_0}{2} \sigma_3 \otimes \mathbf{1} + \omega_0 \mathbf{1}_2 \otimes a^\dagger a + \Omega (\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) \\ &= \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \Omega a \\ \Omega a^\dagger & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix} \end{aligned} \quad (33)$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that  $\rho \in M(2; \mathbf{C}) \otimes M(\mathcal{F}) = M(2; M(\mathcal{F}))$ , where  $M(\mathcal{F})$  is the set of all operators on the Fock space  $\mathcal{F}$ . See for example [12], [13].

Furthermore, it may be possible to treat the generalized master equation given by

$$\begin{aligned} \frac{\partial}{\partial t} \rho &= -i[H_{JC}, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a) \\ &\quad - \frac{\kappa}{2} (a^2 \rho + \rho a^2 - 2a \rho a) - \frac{\bar{\kappa}}{2} ((a^\dagger)^2 \rho + \rho (a^\dagger)^2 - 2a^\dagger \rho a^\dagger) \end{aligned} \quad (34)$$

with the condition  $\mu\nu \geq |\kappa|^2$  similarly in this paper.

These equations ((32), (34)) are very hard to solve in the operator algebra level, so even constructing approximate solutions is not easy. This is our future task, [14].

## References

- [1] K. Fujii : An Approximate Solution of the Master Equation with the Dissipator being a Set of Projectors, arXiv : 0708.4047 [quant-ph].
- [2] R. Endo, K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator, arXiv : 0710.2724 [quant-ph].
- [3] H. -P. Breuer and F. Petruccione : The theory of open quantum systems, Oxford University Press, New York, 2002.
- [4] W. P. Schleich : Quantum Optics in Phase Space, WILEY-VCH, Berlin, 2001.

- [5] K. Fujii : Introduction to Coherent States and Quantum Information Theory, quant-ph/0112090.
- [6] K. Fujii : Matrix Elements of Generalized Coherent Operators, Yokohama Math. J, **53** (2007), 101, quant-ph/0202081.
- [7] R. Alicki, F. Benatti and R. Floreanini : Charge Oscillations in Superconducting Nanodevices Coupled to External Environments, arXiv : 0711.0812 [quant-ph].
- [8] K. Fujii : Quantum Mechanics with Complex Time : A Comment to the Paper by Rajeev, quant-ph/0702148.
- [9] S. G. Rajeev : Dissipative Mechanics Using Complex-Valued Hamiltonians, to appear in Annals of Physics, quant-ph/0701141.
- [10] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime, J. Opt. B : Quantum and Semiclass. Opt, **6** (2004), 502, quant-ph/0407014.
- [11] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime II : Complete Construction of the Controlled-Controlled NOT Gate, Trends in Quantum Computing Research, Susan Shannon (Ed.), **Chapter 8**, Nova Science Publishers, 2006 and Computer Science and Quantum Computing, James E. Stones (Ed.), **Chapter 1**, Nova Science Publishers, 2007, quant-ph/0501046.
- [12] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen : Cavity losses for the dissipative Jaynes-Cummings Hamiltonian beyond Rotating Wave Approximation, J. Phys. A: Math. Theor. **40** (2007), 14527, arXiv:0709.1614 [quant-ph].
- [13] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen : Population trapping due to cavity losses, arXiv:0710.3701 [quant-ph].
- [14] K. Fujii : in progress.